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A generalized Montgomery phase formula for rotating self-deforming bodies

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Abstract

We study the motion of self-deforming bodies with non-zero angular momentum when the changing shape is known as a function of time. The conserved angular momentum with respect to the center of mass, when seen from a rotating frame, describes a curve on a sphere as happens for the rigid body motion, though obeying a more complicated non-autonomous equation. We observe that if, after time ΔT, this curve is simple and closed, the deforming body's orientation in space is fully characterized by an angle or phase θ_M. We also give a reconstruction formula for this angle which generalizes R, Montgomery's well known formula for the rigid body phase. Finally, we apply these techniques to obtain analytical results on the motion of deforming bodies in some concrete examples.

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1. Introduction

1.1. Background

We are going to study the problem of describing the motion of a rotating body whose shape is changing with time in a known controlled fashion. A particular case of this problem is the one in which the body's shape is constant in time, i.e. a *rigid body*.

As is well known, a free *rigid body* rotates about its center of mass in a rather complicated way, depending on how its mass is distributed in space. This distribution is represented by the corresponding *inertia tensor* and the motion is such that the *(spatial) angular momentum* with respect to the *center of mass* is a conserved quantity.

Analytically, the orientation of the body with respect to an inertial reference frame can be obtained by, first, solving *Euler equations* for the (body) angular momentum and, finally, *reconstructing* the desired curve in the space of

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rotations from the momentum one. When the momentum curve completes a period, the orientation of the body in space is the initial one up to a rotation in a certain angle about the (conserved) angular momentum direction. A beautiful result of Montgomery [4] states that this angle, usually called the *rigid body phase*, can be obtained using a *reconstruction formula* (see [1]) which involves a *geometrical* (an holonomy) and a *dynamical* (energy and period values) contribution.

Now, when a rotating body is *free* but not rigid because its shape changes with time in a prescribed fashion, the way in which the mass is distributed in space is thus also changing with time. How does such a body move? Or, since we know how its *shape* is changing: which is the rotation about the center of mass induced by this changing mass distribution? For self-deforming bodies with *zero angular momentum*, this question was answered by Shapere and Wilczek in [8]. In that case, the induced reorientation has a *pure geometric nature* because it is described by a *horizontal curve* with respect to the *mechanical connection* in a $SO(3)$-principal fiber bundle over *shape space* (see [8, 5] and references therein).

Another related problem is that of finding the *optimal sequence of deformations* which induces a *given reorientation* of the deforming body. This is an *optimal control problem* which generalizes the well known *falling cat problem* (see [5]). The problem we want to analyze here is, in a sense, *the problem orthogonal* to the above control problem: we *know* the sequence of deformations and we want to *find* the induced reorientation.

1.2. Main results

In the present paper we shall focus on a case not covered in [8], i.e. the case in which a *self-deforming body* rotates with *non-zero* (conserved) angular momentum.

Our main result is an expression for an angle or *phase* that determines, at specific times, the *exact orientation of a spinning self-deforming body with non-zero angular momentum*, generalizing Montgomery's formula [4].

The examples that we shall be keeping in mind are the ones in which *someone reaccommodates the furniture in a spacecraft* or *a satellite in orbit from which an antenna is coming out*.

Notice that in the above concrete examples, the body is acted on by external forces (e.g. gravity). Nevertheless, also note that for *small* objects like satellites in orbit the angular momentum with respect to the center of mass is approximately conserved. Within this approximation, the full motion can be described by two sets of *decoupled* equations: the ones for the center of mass (a *central force problem*) and the ones we shall give below for the rotation about the center of mass (a *self-deforming body problem*).

The total reorientation of a self-deforming body has two contributions: the one induced from the change in its shape (of *geometric nature* [8]) and the one we shall study, that follows from having non-vanishing angular momentum (of *dynamical nature* as for a rigid body).

In Section 2.2, we define the class of deforming bodies we shall consider, i.e., the one that we shall refer to as *self-deforming* bodies. These are defined by a pure *kinematical constraint* and a *dynamical hypothesis*. In Section 2.3, we shall derive the corresponding set of (second-order) *non-autonomous equations of motion* for the unknown rotation about the center of mass. These follow from the *conservation of the angular momentum* measured from a reference system having its origin at the center of mass and axes parallel to those of an inertial one for all time. We will refer to it as the *spatial angular momentum*.

Also in 2.3, we shall observe that, as in the rigid body problem, the desired induced rotation can be *reconstructed* from a solution of the associated *body angular momentum* (first-order, non-autonomous) equations. This is the angular momentum as seen from a reference frame which is *rotating with* the deforming body (see [8]). At this point, we can re-state our main result: when, after time ΔT, the body angular momentum solution returns to its initial value, the reconstructed rotation curve returns to its initial value up to a rotation about the (conserved) spatial angular momentum direction; moreover, in Section 3 we show that the angle of this rotation or *self-deforming body phase* can be expressed (mod 2π) by the reconstruction formula (13) involving a *geometric* and a *dynamical* term. This result can be seen as a straightforward generalization of Montgomery's formula from the rigid body to the self-deforming body motion.

This formula relates the body's orientation to the (non-conserved) *energy integral* over ΔT and the *geometry* of the (non-zero) body angular momentum solution curve. In the zero-angular-momentum case of [8], as the motion is of a pure geometrical nature, the above phase becomes trivial.

As in the rigid body case, our formula can be applied when we have a geometric description of an underlying simple closed body angular momentum solution curve. Explicit time dependence of the corresponding equations implies that, in general, *energy is not conserved* during the motion of such a body. Also, as the equations for the body angular momentum are non-linear and have generic time dependent coefficients, solutions are hard to describe in the general case.

In view of this last observation, in Section 4 we complete this work by studying some particular classes of deformations. In each case, we shall be able to derive analytical results on the motion of the underlying deforming body by making simple dynamical estimates on the geometry of the body angular momentum solutions and by thus applying the generalized Montgomery formula.

2. Physical setting

2.1. Deformable bodies

Now, we review the setting presented in [5] (see also [8,7]) for deformable bodies.

Let us call Q the *configurations space* of a system of N particles or an extended body from the *center of mass* reference system CM. Thus, $Q = \mathbb{R}^{3N-3}$ or Q is a submanifold of the set of embeddings $\mathbf{q} : B \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ s.t. the center of mass is at the origin, i.e., $M\mathbf{r}_{\mathrm{CM}} = \int_B \mathrm{d}m(x)\, \mathbf{q}(x) = \mathbf{0}$ for B being a reference shape of the extended body, $\mathrm{d}m(x), x \in B$ denoting the corresponding mass density and $M = \int_B \mathrm{d}m(x)$ the total mass. In both cases, the usual action of $SO(3)$ on $\mathbb{R}^3$ gives rise to a natural action of $SO(3)$ on Q. This action turns out to be free on

$$Q_0 = Q - Q_{1D}$$

where Q_{1D} is the set of points in Q representing configurations in which all the particles or the entire body is contained in a straight line. Hence,

$$Q_0 \stackrel{\pi}{\to} Q_0/SO(3)$$

defines a *principal fiber bundle*, whose base $B = Q_0/SO(3)$ is usually called the *shape space*.

In both particle system and extended body cases, the manifold Q_0 (and also Q) has a *Riemannian structure* induced by the usual scalar product of $\mathbb{R}^3$. So there is a natural *principal connection* on the bundle $Q_0 \stackrel{\pi}{\to} Q_0/SO(3)$ defined by choosing as the horizontal subspaces the orthogonal complement to the vertical subspaces with respect to this metric. This is usually called the *mechanical connection* on the bundle $Q_0 \stackrel{\pi}{\to} Q_0/SO(3)$.

Notation. From now on,

- S will denote a given *inertial reference frame*,
- $\mathrm{CM}(t)$ will denote the reference frame with *origin at the center of mass* of the body $r_{\mathrm{CM}}(t)$ for each t and *axes parallel to those of* S,
- $\widetilde{\mathrm{CM}}(t)$ will denote any reference frame with *origin at the center of mass* of the body, with (possibly) *rotating axes with respect to those of* $\mathrm{CM}(t)$.

Remark 2.1 (*Reference Systems*). We can think of a point $q_0 \in Q_0$ over a shape $\pi(q_0) = b_0 \in Q_0/SO(3)$ as giving the configuration of a body with shape represented by b_0 as seen from a reference system CM. Another point $\tilde{q}_0$ s.t. $\pi(\tilde{q}_0) = \pi(q_0)$ then represents the configuration, as seen from CM, of a body with the same shape but, now, rotated with respect to the one represented by q_0. In addition, we can also interpret $\tilde{q}_0$ as describing the *same body* but as seen from a rotated reference system $\widetilde{\mathrm{CM}}$. This last interpretation of the different points of a fiber $\pi^{-1}(b_0)$ is the one that we shall keep in mind for the rest of the paper. See also the discussion in Ref. [8].

2.2. Self-deforming body hypothesis

Let us denote as $\tilde{r}_{io}(t)$ the position at time t of the i-th particle with respect to a (possibly moving) reference frame $\widetilde{S}(t)$. Then, for each time t, there exist a global rotation $R(t) \in SO(3)$ and a translation $T(t) \in \mathbb{R}^3$ such that the position with respect to the inertial reference frame S is

$$r_i = R(t)\tilde{r}_{io}(t) + T(t). \tag{1}$$

A *self-deforming body* is defined to be a system of particles or an extended object satisfying:

(i) *Kinematics*: There exists a *reference frame* $\widetilde{S}(t)$, not necessarily inertial, from which we know $\tilde{r}_{io}(t)$ or, equivalently, a *reference curve* $d_0(t)$ in Q_0 representing the changing shape as seen from $\widetilde{S}(t)$. Consequently, we also have the corresponding *shape space curve* $\tilde{c}(t) = \pi(d_0(t))$.
(ii) *Dynamics*: The constraint forces which act on the particles in order to give this prescribed motions $\tilde{r}_{io}(t)$ are *internal forces* satisfying the *strong action–reaction principle*. This means that all forces acting on the particle i are caused by other particles j and $F_{ij}^{\text{int}} = -F_{ji}^{\text{int}}$ with F_{ij}^{int} parallel to the vector $r_{ij} = r_i - r_j$.

Condition (i) can be seen as a set of *time dependent kinematical constraints* generalizing the usual ones for rigidity: from $\widetilde{S}(t)$ we know how the body's shape is changing (see also [8]).

Example 2.2 (*Spacecraft*). For the system being a spacecraft, $\widetilde{S}(t)$ could be chosen as a frame fixed to some part of the ship or an astronaut.

Remark 2.3 (*Mechanical Forces*). Notice that, although some forces do not satisfy the strong action–reaction principle (for instance, electromagnetic forces), most of mechanical forces do.

Remark 2.4 (*Center of Mass Reference*). We can always take $\widetilde{S}(t) = \widetilde{\text{CM}}(t)$ (recall our notation before Remark 2.1) having its origin at the center of mass at all time. This is, thus, the situation that we shall consider in the rest of this paper. See also the discussion at the end of this section.

Remark 2.5 (*Non-Conservation of Energy*). Note that with such time dependent constraints, the *energy is not conserved* in general because the deformation is implemented by time dependent constraint forces.

The *self-deforming body problem* is that of finding a curve $R(t)$ in $SO(3)$ such that for

$$c(t) = R(t) \cdot d_0(t) \tag{2}$$

in Q_0 the *spatial angular momentum with respect to the c.m. is conserved* (see below). This can also be seen as a *reconstruction problem* (see [1]) for the rotation $R(t)$ from the given $\tilde{c}(t)$.

We end this section with some remarks on the meaning and the measurement of $d_0(t)$. First, we would like to stress that the reference curve $d_0(t)$ is a *natural physical input* for the problem. To illustrate this fact, let us suppose that we want to describe the motion of a spacecraft or satellite when someone is reordering the furniture inside of it, or when an antenna is coming out from this satellite. Before launching, in the lab, an engineer can attach the satellite to the floor and perform exactly the same deformation as will occur in space. The body does not rotate because it is attached, but the position of all its parts can be measured as a function of time t from a lab reference frame. Then, the position of the center of mass can be established for all t and, consequently, the position of every part of the body from a reference system $\widetilde{\text{CM}}(t)$ fixed to the center of mass can be known for each t.

This provides us with a curve $d_0(t)$ as desired: when the satellite is in orbit, the same deformation will occur yielding that $d_0(t)$ projects onto the same curve in shape space as the *physical curve* $c(t)$. Notice that, as the body can freely rotate about its center of mass, the position with respect to $\text{CM}(t)$, represented by $c(t)$, will differ, in general, by a rotation from the one described by $d_0(t)$ for each t. This rotation is precisely the solution $R(t)$ of the *self-deforming body problem*.

Example 2.6 (*Rigid Body*). Note that the rigid body is a special case of the self-deforming body: take $\tilde{r}_{io}(t)$ constant for all time. More generally, $d_0(t)$ must be contained in the fiber over the point representing the constant shape of the rigid body for all t.

2.3. Equations of motion

The equations for $R(t)$, according to our definition of the self-deforming body, can be derived from the *conservation of the angular momentum relative to the center of mass*

$$\overset{\bullet}{L}_{\text{CM}} = 0.$$

This means that the rotation must be such that, from a frame $\text{CM}(t)$ this quantity is conserved even though things are moving internally in the system.

Let us recall the well known quantities: for any $R(t) \in SO(3)$ and $d(t) \equiv \{r_i(t)\} \in Q_0$,

- *body angular velocity* $\omega_B^{R(t)} \simeq R^{-1}\dot{R}$ is defined, as usual, by $\omega_B^{R(t)} \times v = R^{-1}\dot{R}v$ for all $v \in \mathbb{R}^3$; we shall denote as $\Psi : (so(3), [,]) \longrightarrow (\mathbb{R}^3, \times)$ the usual Lie algebra isomorphism (see e.g. [2]);
- (locked) *inertia tensor*: $I : Q_0 \to S^{3\times 3}_{>0} := \{3 \times 3$ real symmetric positive definite matrices $\}$, $v \cdot I(\{r_i\})w = \sum_i m_i (v \times r_i) \cdot (w \times r_i)$;
- *angular momentum* (with respect to a rotated frame with origin at the center of mass): $L : TQ_0 \to \mathbb{R}^3$, $L(\{r_i, \dot{r}_i\}) = \sum_i m_i\, r_i \times \dot{r}_i$, satisfying

$$L\left(\frac{\mathrm{d}}{\mathrm{d}t}(R(t)d(t))\right) = R(t)\, I(d(t))\omega_B^{R(t)} + R(t)\, L\left(\frac{\mathrm{d}}{\mathrm{d}t}(d(t))\right), \tag{3}$$

which gives the *momentum map* for the $SO(3)$ action on TQ_0 (see the details in [1,5]);
- *kinetic energy*: $T : TQ_0 \to \mathbb{R}$, $T(\{r_i, \dot{r}_i\}) = \sum_i m_i\, \dot{r}_i^2$, for which

$$T\left(\frac{\mathrm{d}}{\mathrm{d}t}(R(t)d(t))\right) = \frac{1}{2}\omega_B^{R(t)} \cdot I(d(t))\omega_B^{R(t)} + L\left(\frac{\mathrm{d}}{\mathrm{d}t}(d(t))\right) \cdot \omega_B^{R(t)} + T\left(\frac{\mathrm{d}}{\mathrm{d}t}(d(t))\right). \tag{4}$$

For the *physical* curve $c(t)$ in Q_0, the following quantity must be conserved:

$$L_{\mathrm{CM}} = L\left(\frac{\mathrm{d}}{\mathrm{d}t}c(t)\right) = L\left(\frac{\mathrm{d}}{\mathrm{d}t}(R(t)d(t))\right) = R(t)\, I(d_0(t))\omega_B^{R(t)} + R(t)\, L\left(\frac{\mathrm{d}}{\mathrm{d}t}(d_0(t))\right).$$

Here $I(d_0(t))$ is interpreted as the *inertia tensor measured from the reference frame* $\tilde{S}(t) = \widetilde{\mathrm{CM}}(t)$ and we shall call

$$L_o(t) := L\left(\frac{\mathrm{d}}{\mathrm{d}t}(d_0(t))\right) = \sum_i m_i \widetilde{r}_{io}(t) \times \overset{\bullet}{\widetilde{r}}_{io}(t)$$

the *internal* (or *apparent* [8]) *angular momentum* with respect to $\widetilde{\mathrm{CM}}(t)$.

The (*time dependent, second-order*) *equations of motion* for $R(t)$ thus read

$$\frac{\mathrm{d}}{\mathrm{d}t}L(R(t)d_0(t)) = 0 \tag{5}$$

$$I(d_0(t))\, \overset{\bullet}{\omega}_B = I(d_0(t))\omega_B \times \omega_B + L_o(t) \times \omega_B - \frac{\mathrm{d}}{\mathrm{d}t}(I(d_0(t)))\, \omega_B - \frac{\mathrm{d}}{\mathrm{d}t}L_o(t)$$

when we express them in terms of the body angular velocity ω_B.

The *reconstruction equations* for $R(t)$, once we solved the previous one for ω_B, are

$$\dot{R} = R\, \hat{\omega}_B \tag{6}$$

where $\hat{\omega}_B = \Psi^{-1}(\omega_B)$ with $\Psi : (so(3), [,]) \longrightarrow (\mathbb{R}^3, \times)$ the usual Lie algebra isomorphism (see [2]). The *initial value* $R(t_1)$ must be such that $R(t_1)d_0(t_1) = c(t_1)$ coincides with the initial value of the problem.

Example 2.7 (*Rigid Body*). For the *rigid body*, recall that $d_0(t)$ must be contained on the fiber over a point in shape space. We can then choose the $d_0(t)$ (equivalently $\widetilde{r}_{io}(t)$) to be constant for all t, so $I(d_0(t)) = I$ is constant in time and $L_o = 0$. In this case, we recover the *Euler equations*:

$$I\, \overset{\bullet}{\omega}_B = I\omega_B \times \omega_B$$

as expected.

Also in analogy with the rigid body problem, as $L(\frac{\mathrm{d}}{\mathrm{d}t}c(t)) \in \mathbb{R}^3$ is conserved during the time evolution, if we define

$$\Pi(t) = I(d_0(t))\omega_B^{R(t)} + L\left(\frac{\mathrm{d}}{\mathrm{d}t}d_0\right) \tag{7}$$

we then have that $L(\frac{\mathrm{d}}{\mathrm{d}t}c(t)) = R(t)\Pi(t)$ and, hence, its $\mathbb{R}^3$-norm $\|L(\frac{\mathrm{d}}{\mathrm{d}t}c(t))\| = \|\Pi(t)\|$ is constant for all t. The quantity $\Pi(t)$ represents the *angular momentum measured from the reference frame* $\tilde{S}(t)$ or *body angular momentum*.

Remark 2.8 (*Recovering the Angular Velocity*). Since $I(d_0(t))$ is invertible for all t, we can recover at every time t the angular velocity $\omega_B^{R(t)}$ from $\Pi(t) \in \mathbb{R}^3$:

$$\omega_B^{R(t)} = I^{-1}(d_0(t)) \left(\Pi(t) - L\left(\frac{\mathrm{d}}{\mathrm{d}t} d_0 \right) \right), \quad \forall t. \tag{8}$$

The corresponding *non-autonomous differential equation* for $\Pi(t) \in \mathbb{R}^3$ is

$$\begin{aligned} \dot{\Pi} &= \Pi \times \left(I^{-1}(d_0(t)) \left(\Pi - L\left(\frac{\mathrm{d}}{\mathrm{d}t} d_0(t) \right) \right) \right) \\ \Pi(t_1) &= R^{-1}(t_1) L_{\mathrm{CM}} \end{aligned} \tag{9}$$

whose solutions lie entirely on the *sphere* $S^2_{\|\Pi\|} \subseteq \mathbb{R}^3$ *of radius* $\|L(\frac{\mathrm{d}}{\mathrm{d}t} c(t))\| = \|\Pi(t)\|$.

Using (8), the reconstruction equations for $R(t)$ become

$$\dot{R} = R\ \Psi^{-1} \left(I^{-1}(d_0(t)) \left(\Pi(t) - L\left(\frac{\mathrm{d}}{\mathrm{d}t} d_0 \right) \right) \right) \tag{10}$$

or, equivalently, if we set $R(t_1) = id$ for simplicity

$$R(t) = T \exp \int_{t_1}^{t_2} \mathrm{d}s\ \Psi^{-1} \left(I^{-1}(d_0(s)) \left(\Pi(s) - L\left(\frac{\mathrm{d}}{\mathrm{d}t} d_0(s) \right) \right) \right)$$

where T stands for the time ordered integral (see also [8]).

Remark 2.9 (*Non-Integrability*). In general, as noted before, the explicit time dependence of the self-deforming body tells us that *energy is not conserved* and consequently, we cannot reduce the dimension of the problem any further.

2.3.1. Gauge freedom

By definition, we are given a curve $d_0(t)$ in the configuration space Q_0, but we might want to work with another curve $\tilde{d}_0(t)$ defining an equivalent self-deforming body problem, i.e. $\pi(\tilde{d}_0(t)) = \pi(d_0(t)) = \tilde{c}(t) \in Q_0/SO(3)$. This is the same as describing this self-deforming body from a new reference frame $\tilde{\tilde{S}}(t)$ having the same origin and rotating, in a certain known way, with respect to the initial one $\tilde{S}(t)$ from which the motion, represented by $d_0(t)$, was originally described.

Remark 2.10 (*Gauge Transformations*). This freedom in choosing the initial orientation curve $d_0(t)$ can be seen as *gauge freedom*. Correspondingly, the change $d_0(t) \rightsquigarrow \tilde{d}_0(t)$ can be thought of as a *gauge transformation*. For more details on this analogy, we refer the interested reader to [5,8] and references therein.

Among all possible *lifts* $d_0(t)$ of $\tilde{c}(t)$ we consider two:

(a) the *horizontal lift* with respect to the *mechanical connection* in the bundle $Q_0 \longrightarrow Q_0/SO(3)$; this is equivalent to the problem of finding a lift $\tilde{d}_0(t)$ such that $L(\frac{\mathrm{d}}{\mathrm{d}t}\tilde{d}_0(t)) = 0\ \forall t$ (see also Remark 2.11);

(b) a lift $\tilde{d}_0(t)$ for which the *inertia tensor* $I(\tilde{d}_0(t))$ is *diagonal* for all t; this is equivalent to solving the problem of finding a lift of the base curve $I(d_0(t))$ along the map

$$\begin{aligned} \mathfrak{A} \times SO(3) &\to S^{3\times 3}_{>0} \\ (a, R) &\longmapsto RaR^{-1} \end{aligned}$$

where $\mathfrak{A} := \{3 \times 3$ diagonal positive definite matrices$\}$.

Remark 2.11 (*Deformable Bodies with Zero Angular Momentum*). In Ref. [8], it is shown that, given a shape space curve, the motion of a self-deforming body with zero angular momentum is described by the corresponding horizontal lift as in (a) above. These computations also arise in the *falling cat problem* (see [5,7]) and other interesting problems (see references in [8]).

Remark 2.12 (*Simplifying the Equations*). Choosing a different $d_0(t)$ changes the time dependence of the coefficients of Eq. (9). Thus, an appropriate choice could turn this equation into a *simpler equivalent one*. For example, choosing the horizontal lift implies that the equation has vanishing $L(d_0(t))$ term because this is zero by construction. We also see that there is an obvious simplification when choosing the lift keeping the inertia tensor $I(d_0(t))$ diagonal. But observe that these two simplifications *cannot always be carried out at the same time* since, in general, the horizontal lift does not necessarily diagonalize the inertia tensor.

3. Phases in the self-deforming body motion

3.1. Reconstruction

For completeness, we now describe two kinds of *reconstruction phases* [1] appearing in the configuration space during the motion of the deforming body. In the rest of the paper, we shall focus only on the second (*abelian*) one.

3.1.1. Reconstruction of $c(t)$ from $\tilde{c}(t)$ in the bundle $Q_0 \stackrel{\pi}{\to} Q_0/SO(3)$

Recall that, for each t, both $d_0(t)$ and $c(t)$ belong to the fiber over $\tilde{c}(t)$ in shape space $Q_0/SO(3)$ (see Section 2.2). When the shape space curve is closed in $[t_1, t_2]$, we can then follow the *standard procedure for reconstruction* [1]: choose $d_0(t)$ to be the horizontal lift with respect to the *mechanical connection* having $d_0(t_1) = c(t_1)$ as initial value. Then, $d_0(t_2) = R_G\, c(t_1)$ with R_G being the *holonomy* of the base path $\tilde{c}(t)$ measured from $c(t_1)$ with respect to this connection (see Remark 2.11 and Section 2.1). This is often called the (*non-abelian*) *geometric phase*. Finally, under these assumptions, the reconstruction formula reads

$$c(t_2) = R_D(t_2)\, R_G(t_1)\, c(t_1)$$

where $R_D(t_2)$ is usually called the (non-abelian) *dynamical phase*. This dynamical phase can be obtained by solving Eq. (5) with the initial value $R_D(t_1) = Id$ and with the above *horizontal* choice of $d_0(t)$, i.e. with $L(\dot{d}_0) = 0$. For details on general reconstruction see [1]. The interested reader can find further details about this reconstruction for a deforming body motion with *zero angular momentum* in [8]. For a study of phases in the $N = 3$-body problem, we refer the interested reader to [6].

3.1.2. Reconstruction of $R(t)$ from $\Pi(t)$ in the bundle $SO(3) \longrightarrow S^2_{\|\Pi\|}$

Recall that, in general, the unknown rotation $R(t)$ in Eq. (2) can be reconstructed via (10) once we have solved the Eq. (9) on the sphere. An interesting special case is when this solution $\Pi(t)$ is closed in the interval $[t_1, t_2]$, that is when

$$\Pi(t_1) = \Pi(t_2).$$

In this case, there is a unique angle θ_M naturally associated with this solution and with the initial condition $R(t_1)$ such that

$$R(t_2) = \exp\left(\theta_M \frac{\hat{L}}{\|L\|}\right) R(t_1),$$

yielding

$$c(t_2) = \left[\exp\left(\theta_M \frac{\hat{L}}{\|L\|}\right) R(t_1)\right] d_0(t_2)$$

where $\hat{L} = \Psi^{-1}(L) \in so(3)$. We see that θ_M defines an *abelian reconstruction phase* associated with the initial data $R(t_1)$ (coming from $c(t_1)$). This phase appears when reconstructing $R(t)$ from $\Pi(t)$ in a $U(1)$-principal bundle $SO(3) \longrightarrow S^2_{\|\Pi\|}$ that we shall describe in the next section.

Remark 3.1 (*Interpretation of* θ_M). Recall that $R(t)$ takes the reference frame $\widetilde{\mathrm{CM}}(t)$ to $\mathrm{CM}(t)$. This implies that, at time t_2 as above, the orientation of the body, as seen from $\mathrm{CM}(t)$, is precisely obtained by rotating the known configuration $d_0(t_2)$ about the conserved angular momentum direction (L_{CM}) in the angle θ_M. So this phase *fully characterizes* the position of the deforming body in space at specific times (i.e. t_2).

In the rest of the paper, we shall focus on the latter reconstruction procedure. Note that, as the second phase is abelian, it is more likely to have simpler closed expressions for its reconstruction.

Finally, we note that the most geometrically interesting situation is that in which both the solution $\Pi(t)$ to (9) and the shape space base curve $\tilde{c}(t)$ in $Q_0/SO(3)$ are closed in the same interval $[t_1, t_2]$, i.e.,

$$\Pi(t_1) = \Pi(t_2)$$
$$\tilde{c}(t_1) = \tilde{c}(t_2).$$

When these conditions hold, there is a geometrically defined phase in the bundle $Q_0 \stackrel{\pi}{\to} Q_0/SO(3)$

$$c(t_2) = \Delta R \cdot c(t_1)$$

independent of the choice of $d_0(t)$ (it only depends on the initial value $c(t_1)$) and having the following expression:

$$\Delta R = \exp\left(\theta_M \frac{\hat{L}}{\|L\|}\right) R(t_1)\Delta R_0 R^{-1}(t_1),$$

where $\Delta R_0 = R_0(t_2)R_0^{-1}(t_1)$ and $R(t_1)$ are fixed by the initial condition $c(t_1)$ and the angle θ_M is again given by the *generalized Montgomery formula* presented in the next section.

3.2. Generalized Montgomery formula

In this subsection, we give a *phase formula* for the reconstruction of the rotation $R(t)$ from a closed solution curve $\Pi(t)$ of Eq. (9). This formula generalizes the well known one given by Montgomery in [4] for the rigid body phase. For the proofs, we shall use some differential geometric results that we review below.

3.2.1. Preliminaries

Recall the diagram (see, for instance, [2], p. 438)

$$so^*_-(3) \stackrel{\pi}{\longleftarrow} T^*SO(3) \stackrel{\text{Left}}{\simeq} SO(3) \times so^*(3) \stackrel{J}{\longrightarrow} so^*_-(3)$$
$$\xi \longleftarrow (R, \xi) \longrightarrow Ad^*_{R^{-1}}\xi$$

where: $so^*_-(3)$ denotes the Poisson manifold $so^*(3)$ with its (minus) standard Poisson bracket; π and J are Poisson and anti-Poisson maps respectively and $Ad^*_{R^{-1}}$ denotes the (right) coadjoint action of $SO(3)$ on $so^*(3)$ defined by $\langle Ad^*_R\xi, X\rangle = \langle \xi, Ad_R X\rangle$ for $\xi \in so^*(3)$, $X \in so(3)$. Recall (see e.g. [2]) that J is the *momentum map* associated with the left symplectic action of $SO(3)$ on $T^*SO(3)$. The trivialization $T^*SO(3) \stackrel{\text{Left}}{\simeq} SO(3) \times so^*(3)$ by left translations is known as passing to *body coordinates*.

If we fix an element $L \in so^*_-(3) \simeq so(3) \simeq \mathbb{R}^3$ (the isomorphisms are compatible with the corresponding Poisson brackets), then we have that

$$\Psi(Ad^*_{R^{-1}}\xi) = R\ \Psi(\xi)$$

and thus

$$\pi(J^{-1}(L)) = S^2_{\|\Pi\|}.$$

The sphere $S^2_{\|\Pi\|}$ of radius $\|L\|$ defines a *symplectic leaf* in $so^*_-(3) \simeq so(3) \simeq \mathbb{R}^3$ (see [2]). Moreover, in this case we have that

$$J^{-1}(L) = \{(R, \Pi);\ R \cdot \Pi = L\} \simeq SO(3) \stackrel{\pi}{\longrightarrow} S^2_{\|\Pi\|}$$
$$(R, R^{-1}L) \longmapsto R^{-1}L$$

is a $U(1)$-principal fiber bundle (see [1]).

Now, consider the inclusion $J^{-1}(L) \overset{i}{\hookrightarrow} SO(3) \times so^*(3) \overset{\text{Left}}{\simeq} T^*SO(3)$ and the $u(1)$-valued 1-form on $J^{-1}(L)$

$$A := \frac{1}{\|L\|} i^* \Theta^L \tag{11}$$

where Θ^L is the canonical left invariant 1-form on $T^*SO(3)$ in *body coordinates*. It can be seen that A gives a *principal connection* in the principal $U(1)$-bundle $J^{-1}(L) \xrightarrow{\pi} S^2_{\|\Pi\|}$ ([1]). This connection 1-form satisfies

$$\mathrm{d}A = -\frac{1}{\|L\|} i^* \omega^L$$

where $\omega^L = -\mathrm{d}\Theta^L$ denotes the canonical symplectic 2-form on $T^*SO(3)$ in body coordinates. By the *reduction theorem* ([3], see also [2]),

$$i^* \omega^L = \pi^* \omega_\mu$$

with ω_μ the reduced symplectic form on $S^2_{\|\Pi\|}$. Finally, if $\mathrm{d}S$ denotes the standard area 2-form on the sphere $S^2_{\|\Pi\|} \subseteq \mathbb{R}^3$, then (see [2])

$$\omega_\mu = -\frac{1}{\|L\|} \mathrm{d}S.$$

3.2.2. The formula

With this geometrical background, the following can be easily proved:

Proposition 3.2. *$R(t)$ is a solution of the (second-order) equation of motion (5) iff $(R(t), \Pi(t)) \in J^{-1}(L) \subset T^*SO(3)$ is an integral curve of the time dependent vector field*

$$X(R, \Pi, t) = (R\ \Psi^{-1}(I^{-1}(d_0(t))(\Pi - L(\dot{d}_0))),\ \Pi \times (I^{-1}(d_0(t))(\Pi - L(\dot{d}_0)))).$$

Remark 3.3 (*Hamiltonization*). This result can be viewed as a time dependent *Hamiltonization* from TQ_0 to $T^*SO(3)$ using the momentum map L and, also, a further *reduction* to $J^{-1}(L)$ of the equations of motion problem (5). See also similar comments about reduction for the three-body problem phases in [6].

Thus, reconstructing $R(t)$ from $\Pi(t)$ is the same as finding a curve $(R(t), \Pi(t)) \in J^{-1}(L)$ in the $U(1)$-bundle $J^{-1}(L) \simeq SO(3) \xrightarrow{\pi} S^2_{\|\Pi\|}$ as above such that the projection to the base $\Pi(t) \in S^2_{\|\Pi\|}$ is a solution of (9). Given $\Pi(t)$, we can apply the usual procedure of *reconstruction* [1]: choose $R_0(t) \in J^{-1}(L)$ in a *natural geometric way* as the *horizontal lift* of $\Pi(t) \in S^2_{\|\Pi\|}$ from the initial value $R_0(t_1) = R(t_1)$ with respect to the connection A. Now, let $\theta(t) \in U(1)$ be an angle to be determined by requiring the curve

$$\exp\left(\theta(t)\frac{\hat{L}}{\|L\|}\right) \cdot (R_0(t), R_0^{-1}(t)L) = \left(\exp\left(\theta(t)\frac{\hat{L}}{\|L\|}\right) R_0(t), R_0^{-1}(t)L\right) \in J^{-1}(L) \simeq SO(3)$$

to be the desired integral curve of $X(R, \Pi, t)$. In the above formula, $\hat{L}$ denotes $\Psi^{-1}(L) \in so(3)$.

It follows that $\theta(t)$ must satisfy the following equation:

$$\|L\| \dot{\theta}(t) = I^{-1}(d_0(t))\Pi(t) \cdot \Pi(t) - I^{-1}(d_0(t))L_0(t) \cdot \Pi(t) \tag{12}$$
$$\theta(t_1) = 0.$$

Now, note that if $[t_1, t_2] \subseteq \mathbb{R}$ is a closed interval, and $\Pi : [t_1, t_2] \to S^2_{\|\Pi\|}$ is (any) continuous curve, then its image $\mathrm{Im}(\Pi)$ is a compact, and hence closed, subset of the sphere $S^2_{\|\Pi\|}$. So its complement $\mathrm{Im}(\Pi)^C$ is open and there exists a closed disc $\bar{d} \subseteq \mathrm{Im}(\Pi)^C$. We then have $\mathrm{Im}(\Pi) \subseteq \bar{d}^C$ and we have thus shown:

Lemma 3.4. *The image* $\mathrm{Im}(\Pi)$ *of a continuous map* $\Pi : [t_1, t_2] \to S^2_{\|\Pi\|}$ *is entirely contained in an open disc* $D \subseteq S^2_{\|\Pi\|}$.

We can now state our main result:

Proposition 3.5 (*Generalized Montgomery Formula*). *Let* $\Pi(t)$ *be a solution of* (9) *satisfying that* $\Pi(t_1) = \Pi(t_2)$ *for some interval* $[t_1, t_2]$ *and that the image of* $\Pi : [t_1, t_2] \to S^2_{\|\Pi\|}$ *is a simple closed curve (i.e.* $\mathrm{Im}(\Pi)$ *homeomorphic to the circle* S^1*); then* $R(t_2) = \exp(\theta_M \frac{\hat{L}}{\|L\|})\, R(t_1)$ *and the angle* θ_M *is given* (mod 2π) *by the formula*

$$\theta_M = (\mp)\frac{\mathrm{area}(\tilde{D})}{\|L\|^2} + \frac{1}{\|L\|}\int_{t_1}^{t_2} \mathrm{d}t\ (I^{-1}(d_0(t))\Pi(t) - I^{-1}(d_0(t))L(\dot{d}_0)) \cdot \Pi(t) \tag{13}$$

where $\tilde{D}$ *is a surface in* $S^2_{\|\Pi\|}$ *bounded by the image of* Π. *The* $-$ *(resp.* $+$*) sign corresponds to the case in which the solid angle defined by* $\tilde{D}$ *on the sphere, with its time-oriented boundary* $\Pi(t)$*, is a positive (resp. negative) signed solid angle.*

Remark 3.6 (*Signed Solid Angles*). As usual, we are considering a solid angle in the sphere to be positive or negative by applying the *right hand rule* to its oriented boundary (see [4]). Also notice that, mod 2π, the above formula keeps the same form (i.e., with the $-$ sign) if we replace $\frac{\mathrm{area}(\tilde{D})}{\|L\|^2}$ by the corresponding *signed solid angle*.

Remark 3.7 (*Relation to the Energy*). The integrand in the right hand side of this formula can be expressed in terms of the total kinetic energy (see Eq. (4)):

$$\begin{aligned}&(I^{-1}(d_0)\Pi(t) - I^{-1}(d_0)L(\dot{d}_0)) \cdot \Pi(t)\\ &\quad = 2T\left(\frac{\mathrm{d}}{\mathrm{d}t}(Rd_0)\right) - 2T\left(\frac{\mathrm{d}}{\mathrm{d}t}d_0\right) + I^{-1}(d_0)L(\dot{d}_0)\cdot L(\dot{d}_0) - I^{-1}(d_0)L(\dot{d}_0)\cdot\Pi(t).\end{aligned}$$

Proof. By the above mentioned reconstruction procedure and since $U(1)$ is abelian,

$$\begin{aligned}R(t_2) &= \exp\left(\theta_D\ \frac{\hat{L}}{\|L\|}\right)\cdot\exp\left(\theta_G\ \frac{\hat{L}}{\|L\|}\right)\cdot R(t_1)\\ &= \exp\left(\overbrace{(\theta_D + \theta_G)}^{\theta_M}\ \frac{\hat{L}}{\|L\|}\right)\cdot R(t_1),\end{aligned}$$

where θ_D is the *dynamical phase*, the solution of Eq. (12), and θ_G the *geometric phase*, given by the *holonomy* of the base path $\Pi(t)$ with respect to the connection A and measured from $R(t_1)$. Thus the dynamical contribution θ_D to θ_M is precisely the second term in the r.h.s. of Eq. (13).

Let us then show that the remaining term coincides with the geometric contribution θ_G.

By the hypothesis and Lemma 3.4, $\mathrm{Im}(\Pi)$ is entirely contained in a smooth disc D in $S^2_{\|\Pi\|}$. Since D is contractible, the restricted principal $U(1)$-bundle $J^{-1}(L)\,|_D \longrightarrow D$ is *trivial* and, then, we have a smooth section $s : D \to J^{-1}(L)$. Once we have chosen the disk D containing the curve $\mathrm{Im}(\Pi)$, the existence of a surface $\tilde{D} \subseteq S^2_{\|\Pi\|}$ whose boundary is $\Pi(t)$ is obvious because D is diffeomorphic to an open disk in $\mathbb{R}^2$ and $\mathrm{Im}(\Pi)$ is homeomorphic to S^1. Thus, mod 2π, we can write (see [1])

$$\begin{aligned}\theta_G &= -\int\int_{\tilde{D}} s^*(dA)\\ &= -\frac{1}{\|L\|^2}\int\int_{\tilde{D}}\mathrm{d}S = -\frac{\mathrm{area}(\tilde{D})}{\|L\|^2}\end{aligned}$$

when the solid angle defined by $\tilde{D}$ is positively oriented with respect to the (time-oriented) boundary curve $\Pi(t)$. The last two equalities follow from the results reviewed in the previous section. Formula (13) is then completed. □

Example 3.8 (*Rigid Body*). For the rigid body, the kinetic energy T is conserved and, as we observed previously, $d_0(t)$ can be taken as a point for all t. So $L(\frac{d}{dt}d_0) = 0$ and the inertia tensor $I(d_0) = I$ is constant. In this case, the periodic solutions of Euler equations bound disks on the sphere, and thus $\tilde{D}$ defines the usual signed solid angle and the above formula becomes the well known reconstruction formula derived by Montgomery [4].

4. Some applications

4.1. Solutions on the sphere

We shall now describe some tools which can be used to study the geometry of solutions of Eq. (9) on the sphere $S^2_{\|\Pi\|}$. Focusing on some particular cases we will be able to use this characterization of the solutions to yield analytical results on the motion of self-deforming bodies by calculating the associated generalized Montgomery phase θ_M.

- *Reconstruction of* $R(t)$: When the solution $\Pi(t)$ for some time interval $[t_A, t_B]$ is an open path, we noted before that the rotation $R(t)$ can be expressed as $\exp(\theta(t)\frac{\hat{L}}{\|L\|}) \cdot (R_0(t), R_0^{-1}(t)L)$, with $(R_0(t), R_0^{-1}(t)L)$ the horizontal lift of the base path $\Pi(t)$ with respect to the connection (11) and $\theta(t)$ a solution of Eq. (12). When the solution $\Pi(t)$ is a simple closed curve for a time interval $[t_A, t_B]$, we have a well defined phase θ_M given by formula (13). So, given a solution $\Pi(t)$ in $[t_1, t_2]$, we can find a *total phase* by adding phases corresponding to sub-time intervals $[t_i, t_{i+1}]$ for which the solution is a *simple open arc* in $S^2_{\|\Pi\|}$ or a *simple closed curve* in $S^2_{\|\Pi\|}$. In the first case, we have a phase defined by
$$R(t_{i+1}) = \exp\left(\theta(t_{i+1})\frac{\hat{L}}{\|L\|}\right)\mathrm{Par}(R(t_i))$$
with $\mathrm{Par} : \pi^{-1}(\Pi(t_i)) \longrightarrow \pi^{-1}(\Pi(t_{i+1}))$ the *parallel transport* (see [1]) in the $U(1)$-principal bundle $J^{-1}(L) \xrightarrow{\pi} S^2_{\|\Pi\|}$ of the initial condition $R(t_i)$, and $\theta(t)$ the solution of (12) with $\theta(t_i) = 0$. In the second case, fixing the initial value $R(t_i)$, the phase is defined by $R(t_{i+1}) = \exp(\theta_M \hat{L})\, R(t_i)$ with θ_M given by formula (13).
- *The energy*: As we noted before, in general, the energy is *not a conserved quantity* during the self-deforming body motion. Nevertheless, if we know the evolution of the kinetic energy $T(\frac{d}{dt}(Rd_0))$ with time, we will be able to determine a specific subset of $S^2_{\|\Pi\|}$ in which the corresponding solution $\Pi(t)$ lies. This fact can be shown as follows: let us define for each time t
$$\begin{aligned} E_t &: S^2_{\|\Pi\|} \longrightarrow \mathbb{R} \\ &: \Pi \longmapsto \frac{1}{2}\Pi \cdot I^{-1}(d_0(t))\,\Pi. \end{aligned}$$
Note that
$$E_t(\Pi(t)) = T\left(\frac{d}{dt}(Rd_0)(t)\right) - T\left(\frac{d}{dt}d_0(t)\right) + \frac{1}{2}L(\dot{d}_0(t)) \cdot I^{-1}(d_0(t))L(\dot{d}_0(t))$$
for $\Pi(t)$ a solution of (9). Hence, as $d_0(t)$ is given, $E_t(\Pi(t))$ is uniquely determined by the kinetic energy $T(\frac{d}{dt}(Rd_0)(t))$. In this case, the corresponding solution $\Pi(t)$ on the sphere at time t must lie in the set
$$E_t^{-1}(k(t)) \cap S^2_{\|\Pi\|}$$
where
$$k(t) = T\left(\frac{d}{dt}(Rd_0)(t)\right) - T\left(\frac{d}{dt}d_0(t)\right) + \frac{1}{2}L(d_0(t)) \cdot I^{-1}(d_0(t))L(d_0(t)).$$
The level sets $E_t^{-1}(k(t))$ are (generally rotated) *ellipsoids* for each $k \geqslant 0$ and each t. Also notice that, for a fixed time t_i, the intersection $E_{t_i}^{-1}(k(t_i)) \cap S^2_{\|\Pi\|}$ gives the set where the *body angular momentum of a rigid body with constant inertia tensor equal to* $I(d_0(t_i))$ and *energy* $k(t_i)$ would lie. Finally, the equation for the evolution of $E_t(\Pi(t))$ is
$$\frac{d}{dt}E_t(\Pi(t)) = [\Pi(t) \times I^{-1}(d_0(t))\Pi(t)] \cdot I^{-1}(d_0(t))L(d_0(t)) + \frac{1}{2}\Pi(t) \cdot \frac{d}{dt}[I^{-1}(d_0(t))]\Pi(t)$$
which is coupled to the Eq. (9) for $\Pi(t)$.

- *The arc length*: for a given closed time interval $[t_1, t_2]$ we are going to find a bound for the length of $\Pi([t_1, t_2])$. To that end, we note that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\Pi(t)\right\| = \|\Pi \times (I^{-1}(d_0(t))(\Pi - L(\dot{d}_0(t))))\|$$
$$\leq \|\Pi\|(\|I^{-1}(d_0(t))\Pi\| + \|I^{-1}(d_0(t))L(\dot{d}_0(t))\|)$$

and, if $\|I^{-1}(d_0(t))v\| \leq a^{-1}(t)\|v\|$ for all $v \in \mathbb{R}^3$, $t \in [t_1, t_2]$, then

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\Pi(t)\right\| \leq \|\Pi\|^2 a^{-1}(t) + \|\Pi\| a^{-1}(t)\|L(\dot{d}_0(t))\|.$$

Since $\|\Pi\| = \|L\| = l$ is constant, we then have that

$$\mathrm{length}(\Pi([t_1, t_2])) \leq l \int_{t_1}^{t_2} a^{-1}(t)(l + \|L(\dot{d}_0(t))\|)\mathrm{d}t.$$

When $a^{-1}(t)$ is a very small function (compared to $\frac{1}{l(t_1-t_2)}$), we can deduce that $\Pi([t_1, t_2])$ is contained in a *small patch* in $S^2_{\|\Pi\|}$.

For general time dependent parameters $I^{-1}(d_0(t))$ and $L(\dot{d}_0(t))$ we cannot give a characterization of the solution $\Pi(t)$ of Eq. (9). So we shall focus on some specific cases to illustrate how to handle concrete problems.

4.1.1. *Cases with $I(t) = \mathrm{diag}(I_1(t), I_2(t), I_3(t))$ and $L(\dot{d}_0(t)) = 0$ for all t*

Let us denote by (1, 2, 3) the cartesian axes of $so^*(3) \simeq \mathbb{R}^3$ and, hence, $\Pi = (\Pi_1, \Pi_2, \Pi_3) \in \mathbb{R}^3$. In these cases, the intersections of each axis with the sphere $S^2_{\|\Pi\|}$ give a constant solution of (9), because at that points Π is parallel to $I^{-1}(t)\Pi$ and the r.h.s. of Eq. (9) vanishes.

The equation for the evolution of the energy becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}E_t(\Pi(t)) = \frac{1}{2}\Pi(t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}[I^{-1}(d_0(t))]\Pi(t)$$

and the arc length is bounded by

$$\mathrm{length}(\Pi([t_1, t_2])) \leq l^2 \int_{t_1}^{t_2} a^{-1}(t)\mathrm{d}t.$$

Now, we shall analyze further the special case

$$I_1(t) < I_2(t) < I_3(t)$$

for all $t \in [t_1, t_2]$. Notice that this is the case (up to renumbering the I_i's) for small enough time intervals $[t_1, t_2]$. Under these conditions, the axes of the ellipsoids $E_t^{-1}(k(t))$ coincide with the cartesian axes in $\mathbb{R}^3$ and the arc length is thus bounded by

$$\mathrm{length}(\Pi([t_1, t_2])) \leq l^2 \int_{t_1}^{t_2} I_1^{-1}(t)\mathrm{d}t.$$

Fixing the time t, we have that through each point of $S^2_{\|\Pi\|}$ passes a solution of Euler equations (rigid body) with inertia tensor $\mathrm{diag}(I_1(t), I_2(t), I_3(t))$. For each time we then have the corresponding *homoclinic solutions* (see e.g. [1]), given by the intersection of $S^2_{\|\Pi\|}$ with the ellipsoid of energy $k(t) = \frac{l^2}{I_2(t)}$.

Given a solution $\Pi(t) = (\Pi_1(t), \Pi_2(t), \Pi_3(t))$ of (9) for the interval $[t_1, t_2]$ with initial value $\Pi(t_1)$, the function $f(t) = E_t(\Pi(t))$ reaches a maximum and a minimum on $[t_1, t_2]$, denoted as $E_{\max}$ and $E_{\min}$ respectively. The same happens with the value of the principal moments of inertia $I_i(t)$. The solution $\Pi(t)$ is then contained in a *connected 'crown like' region* R which is the connected component of $\sqcup_{t\in[t_1,t_2]} E_t^{-1}([E_{\min}, E_{\max}]) \cap S^2_{\|\Pi\|}$ which contains the initial value $\Pi(t_1)$.

We can now show the following results on the qualitative behavior of $\Pi(t)$:

1. If $E_{\min} > \frac{l^2}{I_{2_{\min}}}$ then R is contained in either the semi-space $\Pi_1 > 0$ or in $\Pi_1 < 0$. In this case, $\Pi(t)$ evolves in $S^2_{\|\Pi\|}$ describing a trajectory that *orbits surrounding the cartesian axis* 1. More precisely, if the initial point lies in the component with (say) $\Pi_1 > 0$, then the solution will lie in this component for all t in $[t_1, t_2]$. So if we consider spherical coordinates (θ, φ),

   $$\Pi_1 = l\ \cos\theta$$
   $$\Pi_2 = l\ \sin\theta\ \cos\varphi$$
   $$\Pi_3 = l\ \sin\theta\ \sin\varphi$$

   with $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$, it follows that $\theta(t) < \frac{\pi}{2}$ for all t in $[t_1, t_2]$. From Eq. (9) we can deduce that

   $$\frac{\mathrm{d}}{\mathrm{d}t}\varphi = \cos\theta\ [I_2^{-1}(t) - I_1^{-1}(t) + (I_3^{-1}(t) - I_2^{-1}(t))\ \sin^2\varphi] \tag{14}$$

   and thus, since $I_1(t) < I_2(t) < I_3(t)$, $\varphi(t)$ is a *monotonically decreasing* function of time, showing that the solution $\Pi(t)$ tends to describe revolutions about the 1 axis.
2. If $E_{\max} < \frac{l^2}{I_{2_{\max}}}$ then R is contained in either the semi-space $\Pi_3 > 0$ or in $\Pi_3 < 0$. In this case, $\Pi(t)$ evolves in $S^2_{\|\Pi\|}$ describing a trajectory that *orbits around the cartesian axis* 3, as in the previous case.
3. In *other cases*, the solution can pass from *orbiting* around one axis to orbiting around another one. To show this, let us suppose that $E_{t_1}(\Pi(t_1)) > \frac{l^2}{I_{2_{\min}}}$ and that I_1 is constant. Then $\mathrm{length}(\Pi([t_1, t_2])) \leq l^2 I_1^{-1}(t_2 - t_1)$ and so we can choose I_1 such that $\Pi([t_1, t_2])$ is contained in some small patch in $S^2_{\|\Pi\|}$. In the case where I_2 is also constant in time and $I_3(t)$ grows (note that the order is maintained in time), then $\frac{\mathrm{d}}{\mathrm{d}t}E_t(\Pi(t)) = \Pi_3^2(t)\frac{\mathrm{d}}{\mathrm{d}t}I_3^{-1}(t) < 0$. So the energy decreases as fast as we want if we make $I_3(t)$ grow sufficiently fast. Note that $\Pi_3^2(t)$ is bounded from below because $\Pi([t_1, t_2])$ is in a small patch. In this situation, $E_{t_2}(\Pi(t_2))$ can be made smaller than $\frac{l^2}{I_{2_{\min}}}$, so the solution is able to pass from the regime (1) to the regime (2) described above when the energy E_t "*crosses*" the homoclinic energy boundary $\frac{l^2}{I_{2\min}}$.

Remark 4.1 (*Return Time*). In either of the previous cases (1) or (2) we can give a lower bound for the (shortest) return time $\Delta T = t_2 - t_1$ s.t. $\Pi(t_1) = \Pi(t_2)$. If we suppose that the solution starting at $\Pi(t_1)$ satisfies the conditions of (1) above and that it returns to this value for the first time at t_2, then

$$\Delta T = t_2 - t_1 \geq \frac{2\pi\,\|\Pi(t_1) \times (1, 0, 0)\|}{l^2 I_{1_{\max}}^{-1}}.$$

The case corresponding to (2) is analogous.

Now, suppose that we are in the case considered in (1) ((2) is analogous) above and that, in some interval $[t_i, t_{i+1}] \subseteq [t_1, t_2]$, the solution $\Pi(t)$ describes a simple closed curve in $S^2_{\|\Pi\|}$. Then, we can apply formula (13) to find the corresponding phase. Taking into account the time orientation of the closed solution $\Pi(t)$ (fixed by (14)), if $\frac{\mathrm{area}(\tilde{D})}{l^2} < 2\pi$ (resp. $> 2\pi$) we must then take the $+$ (resp. $-$) sign in (13) and we have that

$$\pm\frac{\mathrm{area}(\tilde{D})}{l^2} + \frac{2}{l}\ E_{\min}\ (t_{i+1} - t_i) \leq \theta_M \leq \pm\frac{\mathrm{area}(\tilde{D})}{l^2} + \frac{2}{l}\ E_{\max}\ (t_{i+1} - t_i). \tag{15}$$

Remark 4.2 (*Bounding θ_M mod 2π*). Note that, since θ_M is defined mod 2π, the above bounds yield *non-trivial* information when $\frac{2}{l}\ (E_{\max} - E_{\min})\ (t_{i+1} - t_i) < 2\pi$.

4.2. Examples

We now apply the previous techniques to obtain estimates for the motion of simple classes of deforming bodies.

Example 4.3 (*Global Expansion/Contraction Deformation*). In this case, we suppose that the body is globally shrinking or expanding, that is, the position of a particle from the given reference frame $\tilde{S}$ is

$$r_{i_0}(t) = a(t) r_{i_0}$$

where r_{i_0} is a constant vector and $a(t)$ is a never vanishing positive scale factor. This means that we can choose the curve $d_0(t)$ in Q_0 such that

$$I(d_0(t)) = a^2(t) I_0$$

with I_0 the constant inertia tensor corresponding to the constant configuration $\{r_{i_0}\}$. By a constant rotation, we can choose the reference system $\tilde{S}$ (equivalently, another curve $d_0(t)$) from which I_0 is diagonal. Then, Eq. (9) on the sphere becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \Pi = a^{-2}(t)(\Pi \times I_0^{-1} \Pi).$$

Given an initial value $\Pi(t_1 = 0)$, Eq. (9) can be exactly solved yielding

$$\Pi(t) = \Pi_{\mathrm{RB}} \left(\int_0^t a^{-2}(s)\, \mathrm{d}s \right),$$

with Π_{RB} denoting the rigid body solution of Euler equations $\dot{\Pi} = \Pi \times I_0^{-1} \Pi$ with initial value $\Pi(t_1 = 0)$. Now, the function

$$f(\Pi) = \frac{1}{2} \Pi \cdot I_0^{-1} \Pi$$

is constant along the solutions. Note that $\Pi(t)$ describes a simple closed curve on the sphere for $t \in [0, T]$ when $\int_0^T a^{-2}(t)\, \mathrm{d}t = T_{\mathrm{RB}}$ equals the period of the rigid body solution Π_{RB}. In that case, the corresponding phase is

$$\begin{aligned} \theta_M &= -\Lambda_{\mathrm{RB}} + \frac{2}{\|L\|} f(\Pi) \int_0^T a^{-2}(t)\, \mathrm{d}t \\ &= -\Lambda_{\mathrm{RB}} + \frac{2}{\|L\|} f(\Pi) T_{\mathrm{RB}} \end{aligned}$$

where Λ_{RB} is the (signed) solid angle enclosed by the rigid body periodic solution Π_{RB} with energy $f(\Pi)$. Notice that this phase *coincides with the rigid body phase* for Π_{RB} ([4]). The motion of this kind of uniformly shrinking/expanding or vibrating body is similar to rigid body motion up to a time reparameterization which is induced by the expansion/contraction.

Example 4.4 (*Expansion/Contraction of an Axially Symmetric Body*). Let us consider the case of an axially symmetric body which expands/contracts in the direction of its symmetry axis, i.e., the case in which there exists a curve $d_0(t)$ such that

$$I(d_0(t)) = \mathrm{diag}(I_1(t), I_2, I_3),$$

with $I_2 = I_3$. As in the previous case, Eq. (9) can be exactly solved:

$$\Pi(t) = \Pi_{\mathrm{RB}} \left(\int_0^t \frac{(I_1^{-1}(s) - I_3^{-1})}{I_1^{-1}(0) - I_2^{-1}}\, \mathrm{d}s \right),$$

with Π_{RB} the rigid body solution to the Euler equations $\dot{\Pi} = \Pi \times I^{-1}(d_0(t_1 = 0))\Pi$ and initial value $\Pi(t_1 = 0)$. The function $f(\Pi) = \frac{1}{2} \Pi \cdot I^{-1}(0) \Pi$ is again constant along the solution $\Pi(t)$, which is a simple closed curve for $t \in [0, T]$ when $\int_0^T \frac{(I_1^{-1}(s) - I_3^{-1})}{I_1^{-1}(0) - I_2^{-1}}\, \mathrm{d}s$ equals the rigid body period T_{RB} corresponding to Π_{RB}. In that case, the associated phase is

$$\theta_M = -\Lambda_{\mathrm{RB}} + \frac{1}{\|L\|} \int_0^T \Pi(t) \cdot I^{-1}(d_0(t))\, \Pi(t) \mathrm{d}t,$$

where Λ_{RB} is the (signed) solid angle enclosed by the rigid body periodic solution Π_{RB} with constant energy $f(\Pi)$. Notice that, in general, this phase is *different* from the rigid body phase associated with Π_{RB}.

Example 4.5 (*An Antenna Coming out from a Satellite along a Principal Axis*). We now consider the cases in which

$$I(d_0(t)) = \mathrm{diag}(I_1(t), I_{2_0}, I_{3_0})$$

or

$$I(d_0(t)) = \mathrm{diag}(I_{1_0}, I_{2_0}, I_3(t))$$

with $I_1(t)$ (or $I_3(t)$) an increasing function of time. These cases give simplified models for the situation in which an antenna comes out from an orbiting satellite along one of the principal axes of inertia 1 or 3. Note that the satellite is free to rotate around its center of mass and so its motion can be described by Eq. (8). Suppose that initially $I_1 < I_2 < I_3$. Then, in the first case, as $I_1(t)$ grows this order relation might stop holding after some time, so the solution could pass from orbiting one axis to orbiting another one. Consequently, we have no control on this kind of solution. More precisely,

$$\frac{\mathrm{d}}{\mathrm{d}t} E_t(\Pi(t)) = \frac{1}{2}\Pi_1^2(t)\frac{\mathrm{d}}{\mathrm{d}t}[I_1^{-1}(t)]$$

is negative, implying that the energy decreases and the solution can pass from the case (1) to (2) of the previous section, describing an open curve on the sphere which we cannot characterize in general. In turn, in the second case the ordering prevails and

$$\frac{\mathrm{d}}{\mathrm{d}t} E_t(\Pi(t)) = \frac{1}{2}\Pi_3^2(t)\frac{\mathrm{d}}{\mathrm{d}t}[I_3^{-1}(t)]$$

is also negative. Since the energy decreases, if the initial value $\Pi(t_1)$ corresponds to case (2) of the previous section, the solution also evolves according to (2) and we have a good characterization of its behavior. In particular, if $\Pi([t_i, t_{i+1}])$ is a simple closed curve, we then know that the corresponding reconstructed rotation $R(t_{i+1})$ is $\exp(\theta_M \frac{\hat{L}}{\|L\|})\, R(t_i)$ where from (15),

$$\pm\frac{\mathrm{area}(\tilde{D})}{\|L\|^2} + \frac{2}{\|L\|}\, E_{\min}\,(t_{i+1} - t_i) \le \theta_M \le \pm\frac{\mathrm{area}(\tilde{D})}{\|L\|^2} + \frac{2}{\|L\|}\, E_{\mathrm{initial}}\,(t_{i+1} - t_i),$$

with E_{initial} is the initial (and hence the maximum) value of the energy E_t in $[t_i, t_{i+1}]$ and $E_{\min} = E_{t_{i+1}}$. We thus note that we can have a better description of the motion of the satellite when the *antenna comes out along the largest principal axis of inertia*.

Remark 4.6 (*Slow Deformations*). Intuitively, when the antenna comes out *very slowly*, the motion of the satellite will be close to a rigid body motion. This is reflected in the fact that when $\frac{\mathrm{d}}{\mathrm{d}t}[I_3^{-1}(t)]$ is *very small* (compared to $\frac{l^2 E_{\mathrm{initial}}}{(t_{i+1}-t_i)}$), then $E_{\min} \sim E_{\mathrm{initial}}$ and $\frac{\mathrm{area}(\tilde{D})}{\|L\|^2} \sim \Lambda_{\mathrm{RB}}$. So the phase θ_M is *approximately the same as the rigid body phase* associated with a rigid body with $I = I(d_0(t_i))$ and initial $\Pi(t_i)$.

Remark 4.7 (*Small Bodies in the Gravitational Field*). For a body, v.g. a satellite orbiting the Earth, which is *small* with respect to the distance of interaction with another body (e.g., the Earth), it is a very good approximation to suppose that the gravitational force acting on a particle of mass m_i of this body is

$$F_i = m_i\, \frac{-G\, M}{(r_{\mathrm{CM}} - P)^3}(r_{\mathrm{CM}} - P)$$

where r_{CM} denotes the position of the center of mass of the body. P and M denote the position of the center of mass and the total mass of the second body (e.g., the Earth), respectively. It can be thus easily seen that the equations of motion for the position of the center of mass (a *central force problem*) are totally decoupled from the equations of motion giving the rotation about the center of mass (a *self-deforming body problem* as in Example 4.5).

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